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A FIXED CUBE THEOREM FOR MEDIAN GRAPHS

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The following result is proven: every edge-preserving self-map of a median graph leaves a cube invariant. This extends a fixed edge theorem for trees and parallels a result on invariant simplices in contractible graphs.

1. Introduction

A graph (with a loop at each point) can be regarded as a point set with a reflexive and symmetric relation ("tolerance relation"). From a topological viewpoint, points forming an edge are mutually "close"; accordingly, an edge-preserving function is a kind of "continuous" function. One might expect that "well-behaved" graphs have some fixed point property for such maps. The fixed *point* property, however, is too demanding in this discrete context. Here one is better off with subsets of points that are mutually close, say, such as simplices (complete graphs). The following are known results in this direction:

Theorem A (Nowakowski and Rival [12]). *For every edge-preserving map f on a finite (reflexive) tree there exists an edge E such that $f(E) = E$.*

Theorem B (Poston [14]). *For every edge-preserving map f on a finite contractible (reflexive) graph there exists a simplex S such that $f(S) = S$.*

Notice that Theorem B extends Theorem A. Theorem B was originally presented in a somewhat different setting (viz., fuzzy geometry alias tolerance space), and its proof involved a homology technique. Some part of Theorem B was also obtained by Quilliot [15, 16]. Since [14] is an unpublished dissertation, we will provide a simple and self-contained proof below.

In this paper we will concentrate on *median* (reflexive) *graphs*. Roughly speaking, these are connected graphs in which any three points have a unique *median,* that is: a point which is geodesically in between each two of the given

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points. Median graphs can be regarded as *median algebras* or "convex structures" having strong separation and Helly properties. More details will be given below.

Median graphs (or algebras) have been studied quite extensively in recent years; see Bandelt and Hedlíková [4] for a survey up to 1980, Isbell [6], and Mulder [10]. Convexity aspects are treated in [4, 6], and by van de Vel [17, 18].

Our main result is

Theorem C. *For every edge-preserving map f on a finite (reflexive) median graph there exists a (hyper)cube Q such that $f(Q) = Q$.*

This result also extends Theorem A, but it is independent of Theorem B. A cube of dimension $n \geqslant 2$ is a trivial example of a non-contractible median graph. None of the above theorems can be sharpened to a genuine fixed point theorem: simply consider the two-point simplex and the non-identity involution. More generally, the antipodal involution $x \mapsto x'$ of a cube Q of dimension n does not leave any proper cube (of dimension $<n$) invariant.

In a topological setting, median operators (satisfying weaker conditions) have also been used to study the so-called absolute retracts; see van Mill and van de Vel [7, 8]. It was this analogy which eventually led us to Theorem C.

2. Preliminary results

All groups G in this paper are reflexive (i.e., having a loop at each point) and have no multiple edges. An *induced subgraph* of G is a subset with the induced edge-relation. A *path* P_n of length n in G is a sequence of $n+1$ points $a_0, \ldots, a_n$, where each pair a_i, a_{i+1} forms an edge. A subset H of G is *connected* if each pair of points in H can be joined by a path in H. If G is connected, the distance $d(a, b)$ between two points $a, b \in G$ is the least possible length of a path joining a and b. The notion of an *edge-preserving map* f is used with the obvious meaning. Note that f may collapse edges (between distinct points) since loops are allowed.

A *median graph* is a connected graph with the following property. For each triple of points a_1, a_2, a_3 there is a unique point x such that

$$\forall i \neq j \in \{1, 2, 3\}: \quad d(a_i, x) + d(x, a_j) = d(a_i, a_j).$$

Then x is called the *median* of a_1, a_2, a_3 and is written as $x = m(a, b, c)$. The ternary operation $m: G^3 \to G$ has the following properties, of which the first two are obvious:

(M1) $m(a, b, c) = m(b, a, c) = m(c, b, a)$,

(M2) $m(a, a, b) = a$,

(M3) $m(m(a, b, c), u, v) = m(a, m(b, u, v), m(c, u, v))$.

Ternary algebras satisfying (M1)–(M3) are known as *median algebras*. Let (X, m) be a *median algebra*. A subset $C \subseteq X$ is called *convex* provided that $m(C \times C \times X) \subseteq C$. For instance, the *interval* between $a, b \in X$,

$$I(a, b) = \{x \mid m(a, b, x) = x\},$$

is convex, as one can easily deduce from the axioms. For a median graph, this set equals the set of all points on geodesics from a to b, that is:

$$I(a, b) = \{x \mid d(a, x) + d(x, b) = d(a, b)\}.$$

Furthermore, intersection and updirected union of convex sets yield convex sets again. This "convex structure" on X has two important features:

(C1) (Nieminen [11]). If C is convex and if $x \notin C$, then there is a convex set H with a convex complement (i.e., a *half-space*) such that $C \subseteq H$ and $x \notin H$.

(C2) (Isbell [6]). If $C_1, \ldots, C_n$ are pairwise intersecting convex sets, then $\bigcap_{i=1}^n C_i \neq \emptyset$.

From (C1) and (C2) one can derive the following separation property (see [18], cf. [10] for the finite case):

(C1′) If C, D are disjoint convex sets, then there is a half-space H with $C \subseteq H$ and $D \cap H = \emptyset$.

We are now ready to state and prove some auxiliary results.

Lemma 1. *In a median graph, the median operator is edge-preserving in each variable separately.*

Proof. Let G be a median graph with $a, b \in G$ fixed. Consider the function

$$f: G \to G, \quad f(x) = m(x, a, b).$$

From the axiom (M3) it follows that for all u, v, w in G,

$$m(u, f(v), f(w)) = f(m(u, v, w)).$$

Hence by [4, Proposition 5.1] f maps convex sets to convex sets. Suppose that $x, y \in G$ are neighbours (i.e., form an edge). Then, with the above notation, $I(x, y) = \{x, y\}$. Therefore $\{f(x), f(y)\}$ is convex and hence is an edge. □

We recall the following concept. Let (X, m) be a median algebra, and let $Y \subseteq X$ be a subset. Then the *subalgebra generated* by Y is the smallest set $\hat{Y} \subseteq X$ such that

$$Y \subseteq \hat{Y}, \qquad m(\hat{Y}^3) \subseteq \hat{Y}.$$

Lemma 2. *Let G be a median graph and let $H \subseteq G$ be a connected subset. Then the median algebra $\hat{H}$ generated by H is again connected, and hence $\hat{H}$ is a median graph (as an induced subgraph).*

Proof. Let $H_0 = H$ and, inductively, $H_{n+1} = m(H_n^3)$. Clearly, $\hat{H} = \bigcup_n H_n$. By assumption, H_0 is connected. Suppose H_n is connected, and let $x = m(a, b, c)$, where $a, b, c \in H_n$. To see that H_{n+1} is connected, it suffices to construct a path from x to a. To this end, choose a path $c_0 = c, c_1, \ldots, c_k = a$, in H_n. Then put

$$x_0 = x, \qquad x_i = m(a, b, c_i) \quad (i = 1, \ldots, k).$$

By Lemma 1, this is a path, which by construction lies within H_{n+1}, and joins $x_0 = x$ with $x_k = a$. $\square$

Let (X, m) be a finite median algebra, and let π be a real-valued function on X with $\pi(x) > 0$ for each $x \in X$. We regard π as a *positive weight function*. For a subset $Y \subseteq X$ we put

$$\pi(Y) = \sum_{y \in Y} \pi(y) \quad \text{(the weight of } Y\text{)}.$$

Then the *median of X with respect to π* is the set

$$\operatorname{Med}(X, \pi) = \bigcap \{H \mid H \subseteq X \text{ a half-space}; \pi(H) > \pi(X - H)\}.$$

This terminology is justified by the results in [3].

Lemma 3. *For a finite median algebra (X, m) and for a positive weight function π, the median of X with respect to π is a cube.*

Proof. If $Y_1, Y_2 \subseteq X$ are subsets such that

$$\pi(Y_1) > \pi(X - Y_1) \quad \text{and} \quad \pi(Y_2) > \pi(X - Y_2),$$

then obviously $Y_1 \cap Y_2 \neq \emptyset$. Applying this to half-spaces and using the Helly property (C2), we infer that $Q = \operatorname{Med}(X, \pi)$ is a nonempty convex set. To see that Q is a cube, we use the following result of [6] (see also [1, 17]): a finite median algebra is a cube provided that any two disjoint half-spaces form a cover.

Let $H_1, H_2 \subseteq Q$ be disjoint (relative) half-spaces. We may assume that both are nonempty sets. Since disjoint convex sets in X (such as H_1, $Q - H_1$ or H_2, $Q - H_2$) extend to complementary half-spaces by (C1′), we obtain two half-spaces $\bar{H}_1, \bar{H}_2$ of X with

$$\bar{H}_1 \cap Q = H_1, \qquad \bar{H}_2 \cap Q = H_2.$$

Now, $\bar{H}_1$ ($\bar{H}_2$, resp.) contains at least one point of Q and misses another one, so that

$$\pi(\bar{H}_1) = \tfrac{1}{2}\pi(X), \qquad \pi(\bar{H}_2) = \tfrac{1}{2}\pi(X).$$

Then $\bar{H}_1 \cap \bar{H}_2 = \emptyset$, for otherwise,

$$H_1 \cap H_2 = \bar{H}_1 \cap \bar{H}_2 \cap Q \neq \emptyset,$$

by the Helly property (C_2). Hence as each point of X has a positive weight we find that $\bar{H}_1 \cup \bar{H}_2 = X$, yielding $H_1 \cup H_2 = Q$. $\square$

In addition to Lemma 3, we note from [10] or [17] that every cube in a median graph is a convex set.

3. Proof of Theorem C

Let G be a finite median graph, and let $f: G \to G$ be an edge-preserving map. Let $x \in G$ minimize the number $d(x, f(x))$. Since f does not increase distances, we get for all n that

$$d(f^n(x), f^{n+1}(x)) = d(x, f(x)).$$

Since G is finite, the sequence $f^n(x)$, $n \geqslant 0$, has a period k. So, without loss of generality, $f^k(x) = x$.

Choose a shortest path P connecting x with $f(x)$. Then $f^k(P)$ is another path connecting these points. As there are only finitely many possibilities for such a path, we may assume that for some $l \geqslant 1$, $f^{kl}(P) = P$. Then put

$$Y = \bigcup_{j=1}^{kl} f^j(P).$$

Note that Y is connected, and that

(1) $f(Y) \subseteq Y$,

(2) f^{kl} is the identity map on Y.

Indeed, since $f^{kl}(P) = P$ and f^{kl} preserves edges, we see that f^{kl} is the identity map on P, whence (2) follows.

We next consider the median subalgebra $\hat{Y}$ generated by Y:

$$Y_0 = Y, \qquad Y_{n+1} = m(Y_n^3), \qquad \hat{Y} = \bigcup_n Y_n.$$

Assume that for some n

(3) $f(Y_n) \subseteq Y_n$,

(4) f^{kl} is the identity map on Y_n.

Take $x \in Y_{n+1}$, say: $x = m(a_1, a_2, a_3)$ with $a_1, a_2, a_3 \in Y_n$. Then

(5) $\forall i \neq j \in \{1, 2, 3\}$: $\quad d(a_i, x) + d(x, a_j) = d(a_i, a_j)$.

By (4), and since f^{kl} preserves edges, we get

$$\begin{aligned} d(a_i, f^{kl}(x)) + d(f^{kl}(x), a_j) &= d(f^{kl}(a_i), f^{kl}(x)) + d(f^{kl}(x), f^{kl}(a_j)) \\ &\leqslant d(a_i x) + d(x, a_j) \\ &= d(a_i, a_j) = d(f^{kl}(a_i), f^{kl}(a_j)), \end{aligned}$$

whence by the triangle inequality for d, we have equalities throughout. By definition, it follows that

$$f^{kl}(x) = m(a_1, a_2, a_3) = x.$$

Having shown that f^{kl} is the identity on Y_{n+1} we infer that f preserves the distance between points of Y_{n+1}. Using (5) again, we obtain

$$d(f(a_i), f(x)) + d(f(x), f(a_j)) = d(f(a_i), f(a_j)),$$

whence

$$f(x) = m(f(a_1), f(a_2), f(a_3)),$$

which is in Y_{n+1} by (3). The induction is now completed, and we conclude that $f(\hat{Y}) = \hat{Y}$. Note from the argument above that f is also median-preserving on $\hat{Y}$, and note from Lemma 2 that $\hat{Y}$ is a median graph.

The median graph $\hat{Y}$ is endowed with the natural convexity described earlier. Since f is a bijection of $\hat{Y}$ preserving the median, it is an automorphism for the related convexity. In particular, f maps half-spaces to half-spaces. Consider the "standard" weight function $\pi(y) = 1$, $y \in \hat{Y}$. Clearly, f maps the median $Q = \mathrm{Med}(\hat{Y}, \pi)$ of $\hat{Y}$ with respect to π onto itself. By Lemma 3, this yields the desired invariant cube Q. □

A consequence of this proof (not of the theorem) is the following: if the number $\#G$ of points is odd and if f is an automorphism of G, then f has a fixed point. Indeed, with the above notation, $\mathrm{Med}(G, \pi)$ is left invariant by f. Clearly, no half-space H of G satisfies $\pi(H) = \frac{1}{2}\pi(G)$, and by inspecting the proof of Lemma 3, we obtain from (C1) that $\mathrm{Med}(G, \pi)$ must be a singleton (cf. [3]).

4. Proof of Theorem B

If G is a graph and if $P_n = \{0, 1, \ldots, n\}$ is a path, then the product set $G \times P_n$ can be organized as a graph (viz., the relational product of the reflexive graphs G and P_n) in which (x, i) and (y, j) form and edge if and only if x, y form an edge in G and $|i - j| \leq 1$. Now, G is called *contractible* if there exists an edge-preserving map (*contraction*)

$$F : G \times P_n \to G,$$

and a point $a \in G$ with

$$F(x, 0) = x, \qquad F(x, n) = a \quad \text{for all } x \in G.$$

Henceforth let G be a finite contractible graph. One easily verifies that, if G has more than one point, then

(1) there exist two adjacent points x, y such that each neighbour of x is also a neighbour of y.

If H is another graph and if $h : H \to G$ is edge-preserving, then the map

$$\tilde{h} : H \to G, \qquad \tilde{h}(u) = \begin{cases} h(u), & \text{if } h(u) \neq x, \\ y, & \text{if } h(u) = x, \end{cases}$$

is again edge-preserving by the choice of x, y; see (1). Applying this to a contraction of G, we find:

(2) $G - \{x\}$ is also contractible.

It is also easy to see that, conversely, (1) and (2) guarantee that the given graph is contractible; cf. Quilliot [15].

We now establish Theorem B. Let $f: G \to G$ be edge-preserving. We prove by induction on $\#G$ that f fixes a simplex (i.e., complete subgraph) of G. Define a map $\bar{f}$ on G as above, and restrict it to a map from $G - \{x\}$ to $G - \{x\}$. By (2) and by assumption, there is a simplex $\bar{S} \subseteq G - \{x\}$ with $\bar{f}(\bar{S}) = \bar{S}$. If $\bar{f} = f$ on $\bar{S}$, then we are done. So assume that $\bar{f} \neq f$ on $\bar{S}$. Then for some $z \in \bar{S}$, we have $f(z) = x$ and hence $\bar{f}(z) = y \in \bar{S}$. We argue by induction: assume $n \geqslant 0$ and

$$\bar{S} \cup \{f^k(z) \mid 0 \leqslant k \leqslant n\} \quad (f^0(z) = z)$$

is a simplex in G. Then $f^n(z)$ is a neighbour of z, and hence $f^{n+1}(z)$ is a neighbour of $f(z) = x$. By (1), $f^{n+1}(z)$ is a neighbour of y. Take $u \in \bar{S}$, $u \neq y$. Let $v \in \bar{S}$ with $\bar{f}(v) = u$. Then $u = \bar{f}(v) = f(v)$, and as v is adjacent to $f^n(z)$, we get that u is adjacent to $f^{n+1}(z)$. Thus, $f^{n+1}(z)$ is adjacent to all members of $\bar{S}$. In particular, $z = f^0(z)$ is adjacent to $f^{n+1}(z)$, and for $1 \leqslant k \leqslant n$ we have that $f^k(z)$ and $f^{n+1}(z)$ are adjacent since $f^{k-1}(z)$ and $f^n(z)$ are. We conclude that

$$\bar{S} \cup \{f^k(z) \mid 0 \leqslant k \leqslant n+1\}$$

is also a simplex of G. The sequence $f^n(z)$, $n \geqslant 0$, must have some period $k \geqslant 0$, and the desired simplex is then

$$S = \{f^n(z) \mid l \leqslant n < l + k\}, \quad \text{for some } l \geqslant 0. \quad \square$$

Nowakowski and Winkler [13] and Quilliot [15] have obtained a game-theoretical characterization of finite contractible graphs.

5. Concluding remarks and problems

5.1. The similarity of Theorems B and C suggests the following problem. Is there a "fixed subgraph" theorem for other classes of graphs, where the fixed objects are either small in size or diameter, or highly symmetric, or kind of "building blocks" for the class of graphs under consideration?

5.2. The analogy of Theorem C and a particular instance of Theorem B can be made more precise. A *retract* of a (reflexive) graph G is the image of an edge-preserving idempotent self-map of G. The median graphs are precisely the retracts of cubes; see Bandelt [2]. A cube of dimension n is the Cartesian product of n copies of the two-point simplex (that is, the path of length 1). On the other hand, the Helly graphs in the sense of Quilliot [15] are contractible, and they are precisely the retracts of relational products of finite paths (see Misane [9] for a survey on these matters). Now, for any class $\mathcal{K}$ of finite graphs, $\mathcal{K}$ and the class

$R(\mathscr{K})$ of all retracts of graphs in $\mathscr{K}$ give rise to the same subclass

$$\text{Fix}\,\mathscr{K} = \text{Fix}\,R(\mathscr{K}),$$

of (minimal) fixed subgraphs under edge-preserving maps. It would be interesting to see further examples, where Fix commutes with some product P, i.e.: for which classes $\mathscr{K}$ it is true that

$$\text{Fix}\,P(\mathscr{K}) = P(\text{Fix}\,\mathscr{K})?$$

At least, this holds for the class $\mathscr{K}$ of finite paths and the relational product or the Cartesian product by virtue of Theorem B and Theorem C, respectively.

5.3. Nowakowski and Rival [12] actually proved that a graph has the fixed edge property if and only if it is a tree without infinite paths. This may suggest an extension of Theorems B, C to infinite graphs, where, in addition, invariant infinite paths have to be considered. In the infinite case, of course, "fixed subgraph" must be substituted by "invariant subgraph".

5.4. A well-known theorem of Kakutani (see e.g. [5]) asserts that a compact convex set in a locally convex vector space has the fixed points property for suitably continuous, convex-valued multi-functions. The topological theory of median algebras with their intrinsic convexity illustrates that these structures are equally well-behaved (in fact, an adapted form of Kakutani theorem is valid there); see van de Vel [19]. In the discrete case, the following question is natural. Let G be a finite median graph, and let F be a convex-valued transformation of G into itself, which is edge-preserving in the following sense: if $x_1, x_2 \in G$ are neighbours, then each y_1 in $F(x_1)$ is adjacent to some y_2 in $F(x_2)$, and vice versa. Does there exist a cube $Q \subseteq G$ with $F(Q) \cap Q \neq \emptyset$? Or with $F(x) \cap Q \neq \emptyset$ for each $x \in Q$?

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